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AN EXPLORATION OF LOCALLY PROJECTIVE POLYTOPES

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This article completes the classification of finite universal locally projective regular abstract polytopes, by summarising (with careful references) previously published results on the topic, and resolving the few cases that do not appear in the literature. In rank 4, all quotients of the locally projective polytopes are also noted. In addition, the article almost completes the classification of the infinite universal locally projective polytopes, except for the $\{\{5,3,3\},\{3,3,5\}_{15}\}$ and its dual. It is shown that this polytope cannot be finite, but its existence is not established. The most remarkable feature of the classification is that a nondegenerate universal locally projective polytope $\mathcal{P}$ is infinite if and only if the rank of $\mathcal{P}$ is 5 and the facets of $\mathcal{P}$ or its dual are the hemi-120-cell $\{5,3,3\}_{15}$.

1. Introduction

The classical study of polytopes concentrated on what are now called "spherical" polytopes, that is, polytopes which were topologically tesselations of n-dimensional spheres. The modern "abstract" polytopes include tesselations of euclidean and hyperbolic space and other spaceforms, as well as objects that are best described by their "local" topology, that is, the topology of their facets and vertex figures, or smaller sections. Good accounts of the classical theory and of the emergence of the theory of the abstract polytopes are to be found in [1] or [14]. Essentially, an abstract n-polytope is a partially ordered set $\mathcal{P}$ with unique maximal and minimal elements, and an order-preserving map from $\mathcal{P}$ onto $\{-1,0,\dots,n-1,n\}$. The rank of $\mathcal{P}$ is n, and the rank of an element of $\mathcal{P}$ is its image under this map. A *section* f/g

Mathematics Subject Classification (2000): 51M20, 52B15 of $\mathcal{P}$ is the set $\{h \in \mathcal{P} : f \geq h \geq g\}$. If the rank of f is two more than that of g, we require that the section f/g contain only two elements besides f and g. A connectivity condition is also required – see [14] or other references for details. The facets of $\mathcal{P}$ are those sections where f has rank $n-1$ and g is the minimal element. Similarly, the vertex figures of $\mathcal{P}$ are those where f is the maximal element and g has rank 0. A section of an n-polytope will be a k-polytope for some $k \leq n$. In particular, the facets and vertex figures of an n-polytope will be $(n-1)$-polytopes (that is, polytopes of rank $n-1$). The dual of $\mathcal{P}$ is the polytope with the same elements as $\mathcal{P}$, but with the order reversed.

A polytope is defined to be *regular* if its automorphism group acts transitively on the set of its *flags*, or maximal totally ordered subsets. The bulk of the study of abstract polytopes has concerned the regular case, although some weaker forms of "regularity" have been also been examined. For example, a polytope is called *section regular* if f/g is isomorphic to f'/g' whenever $\operatorname{rank} f = \operatorname{rank} f'$ and $\operatorname{rank} g = \operatorname{rank} g'$. A polytope is *equivelar* if the latter holds merely for the sections of rank 2. All regular polytopes are section regular, and all section regular polytopes are equivelar.

A polytope has *Schläfli type* or *Schläfli symbol* $\{p_1, \ldots, p_{n-1}\}$ if the sections f/g for $\operatorname{rank} f - 1 = i = \operatorname{rank} g + 2$ are p_i-gons. The Schläfli type is well-defined if and only if the polytope is equivelar. An equivelar polytope is *degenerate* if its Schläfli type contains a 2. Otherwise, it is *non-degenerate.*

Given a polytope $\mathcal{P}$, and a subgroup N of its automorphism group, we may form the *quotient* $\mathcal{P}/N$ of $\mathcal{P}$ by N by taking as elements the orbits of elements of $\mathcal{P}$ under the automorphism action of N, and setting $f \cdot N \geq g \cdot N$ for every f and g with $f \geq g$. This will not always be a polytope. In the case that it is, we say $\mathcal{P}$ *covers*, or is a *cover* for, its quotient $\mathcal{P}/N$. If $\mathcal{P}$ is regular, the group N may be chosen to be a *semisparse subgroup* of the automorphism group of $\mathcal{P}$. A *semisparse subgroup* N of $\operatorname{Aut}(\mathcal{P})$ is one for which $\mathcal{P}/N$ is a polytope, and which stabilises a flag of $\mathcal{P}/N$ under a certain action. See [9] for more information.

An important result in the theory of abstract polytopes is that any section regular polytope $\mathcal{Q}$ whose facets are isomorphic to $\mathcal{K}$, and whose vertex figures are isomorphic to $\mathcal{L}$, where $\mathcal{K}$ and $\mathcal{L}$ are regular, may be expressed as a quotient of a *universal* such polytope, which is denoted $\{\mathcal{K}, \mathcal{L}\}$ (Theorem 2.5 of [9], or Theorem 4A2 of [14] in the case $\mathcal{Q}$ is regular). The presentation of the automorphism group of this universal polytope is easily derived from those of $\operatorname{Aut}(\mathcal{K})$ and $\operatorname{Aut}(\mathcal{L})$ (see Proposition 4A8 of [14]).

Extensive study has been made by McMullen, Schulte and others of the so-called *locally toroidal* polytopes (definition follows). See for example

Chapters 10 to 12 of [14], which bring together and update earlier work on the topic (such as [18]). Comparatively little has been done, until recently, on polytopes of other local topological types. The next three paragraphs summarise briefly the literature on locally projective polytopes.

There are two non-equivalent definitions used in the literature for a "locally X" polytope. The broader definition is that the minimal nonspherical sections of the polytope must be of topological type X (see for example [13]). The narrower definition is that these minimal nonspherical sections must actually be the facets or vertex figures (or both) of the polytope (see for example [18]).

In this article, the narrower definition is followed. For rank 4 polytopes, both definitions are equivalent: there do not exist nonspherical polytopes of rank less than 3, so the minimal nonspherical sections of a rank 4 polytope must be of rank 3, if they exist at all. The terminology "locally X" is normally restricted to regular polytopes, or at least to section regular polytopes.

In the late 1970s and early 1980s, Grünbaum and Coxeter independently discovered a self-dual polytope with 11 hemi-icosahedral facets (see [3] and [5]). During the same period (see [2]), Coxeter discovered another self-dual polytope with 57 hemidodecahedral facets.

As the theory of abstract polytopes developed, these sporadic examples of locally projective polytopes were joined by others that arose out of more general considerations. Schulte, in [17], noted some examples of locally projective polytopes of type $\{4,3,4\}$, that is, with hemicubes for facets, and octahedra or hemioctahedra for vertex figures. Some years later, in 1994, McMullen in [13] examined polytopes that were locally projective in the broader sense, of type $\{3^{k-2},4,3,3,4,3^{n-k-3}\}$ (that is, $\{3,4,3^{n-3}\}$, $\{3,3,4,3^{n-4}\}$, $\{4,3,3,4,3^{n-5}\}$ respectively for $k=0,1,2$), where $n \geq 5$. This was the first paper to give an infinite family of polytopes which were "locally projective" in some sense. This paper now appears as a chapter of [14]. As was shown in [8], the theory developed by Schulte in [17] can be easily applied to produce another infinite family. However, Schulte had chosen a different focus for that paper.

The most recent work on the problem has been a series of articles by the author and others (see [7], [8] and [10]) which focused on specific examples of locally projective polytopes and their quotients. That work almost completed the classification of locally projective rank 4 polytopes and their quotients, and made significant inroads in higher ranks.

This article completes the classification of finite universal polytopes that are locally projective in the stronger (that is, narrower) sense, that is, finite universal polytopes that have either spherical or projective facets and vertex figures, not both spherical.

One remarkable feature of the classification is the fact that all such polytopes are finite, except in rank 5. As shall be seen, this is not so surprising in rank 6 or higher, but it is a surprise in rank 4.

2. The Form and Scope of the Classification

The usual way to search for polytopes with given (regular) facets $\mathcal{K}$ and vertex figures $\mathcal{L}$ is to find the *universal* such polytope $\mathcal{P}=\{\mathcal{K},\mathcal{L}\}$, and then seek its quotients. The universal polytope is characterised by its automorphism group $\mathrm{Aut}(\mathcal{P})=\Gamma(\mathcal{P})=[\mathcal{K},\mathcal{L}]$.

The facets and vertex figures of a locally projective section regular polytope of rank n are either spherical or projective polytopes of rank $n-1$. Therefore, the most straightforward strategy to classify these polytopes is as follows.

- Identify the projective and spherical rank $n-1$ polytopes.
- For each pair of spherical or projective $\mathcal{K}$ and $\mathcal{L}$, not both spherical, identify the universal $\{\mathcal{K},\mathcal{L}\}$, if it exists.

As an aside, if we were seeking polytopes that were locally projective in the broader sense, we would allow $\mathcal{K}$ and $\mathcal{L}$ to be locally projective, not just spherical or projective. It will be noted in Section 8 that this would be a vastly more difficult classification problem.

Note for the second step, if $\mathcal{L}$ is of type $\{p,q_2,\dots,q_{n-2}\}$ and $\mathcal{K}$ of type $\{q'_2,\dots,q'_{n-2},r\}$, a simple necessary condition that $\{\mathcal{K},\mathcal{L}\}$ exist is that q_i must equal q'_i for all i. Then $\{\mathcal{K},\mathcal{L}\}$ will be of type $\{p,q_2,\dots,q_{n-2},r\}$. As we shall see, this necessary condition is not sufficient. If any $q_i=2$, there can be no locally projective polytopes, since the facets and vertex figures would be degenerate, and there are no degenerate projective polytopes. There do exist locally projective polytopes with $p=2$ or $r=2$. The only cases for $r=2$ are "ditopes" $\{p,q_2,\dots,q_{n-2},2\}$ with projective facets of type $\{p,q_2,\dots,q_{n-2}\}$. Their enumeration is easy. If $p=2$ we obtain the duals of the ditopes. For most of the rest of this article, we shall ignore these degenerate cases and assume $p,q_i,r\geq 3$.

We now consider in particular locally projective polytopes of rank 4. The spherical rank 3 polytopes have been known since the time of the ancient Greeks. They are the tetrahedron $\{3,3\}$, the cube $\{4,3\}$, the octahedron or cross-polytope $\{3,4\}$, the dodecahedron $\{5,3\}$ and the icosahedron $\{3,5\}$. Except for the tetrahedron, each of their symmetry groups has a central inversion ω. Taking the quotient of these polytopes by the corresponding groups $\langle\omega\rangle$ yields four projective polytopes: the hemicube $\{4,3\}_3 = \{4,3\}/2$, its dual the hemicross, the hemidodecahedron $\{5,3\}_5 = \{5,3\}/2$, and its dual the hemi-icosahedron. It is well known, as noted in [6], that these are the only non-degenerate proper quotients of the platonic solids[1], and therefore, the only rank 3 section regular projective polytopes.

Ignoring duality leads to 22 cases that need to be analysed. These are listed in Table 1.

#	Facets	V.Figures	Type	#	Facets	V.Figures	Type
1	$\{3,3\}$	$\{3,4\}_3$	$\{3,3,4\}$	2	$\{3,3\}$	$\{3,5\}_5$	$\{3,3,5\}$
3	$\{3,4\}$	$\{4,3\}_3$	$\{3,4,3\}$	4	$\{3,4\}_3$	$\{4,3\}_3$	$\{3,4,3\}$
5	$\{3,4\}_3$	$\{4,3\}$	$\{3,4,3\}$	6	$\{3,5\}$	$\{5,3\}_5$	$\{3,5,3\}$
7	$\{3,5\}_5$	$\{5,3\}_5$	$\{3,5,3\}$	8	$\{3,5\}_5$	$\{5,3\}$	$\{3,5,3\}$
9	$\{4,3\}_3$	$\{3,3\}$	$\{4,3,3\}$	10	$\{4,3\}$	$\{3,4\}_3$	$\{4,3,4\}$
11	$\{4,3\}_3$	$\{3,4\}_3$	$\{4,3,4\}$	12	$\{4,3\}_3$	$\{3,4\}$	$\{4,3,4\}$
13	$\{4,3\}$	$\{3,5\}_5$	$\{4,3,5\}$	14	$\{4,3\}_3$	$\{3,5\}_5$	$\{4,3,5\}$
15	$\{4,3\}_3$	$\{3,5\}$	$\{4,3,5\}$	16	$\{5,3\}_5$	$\{3,3\}$	$\{5,3,3\}$
17	$\{5,3\}$	$\{3,4\}_3$	$\{5,3,4\}$	18	$\{5,3\}_5$	$\{3,4\}_3$	$\{5,3,4\}$
19	$\{5,3\}_5$	$\{3,4\}$	$\{5,3,4\}$	20	$\{5,3\}$	$\{3,5\}_5$	$\{5,3,5\}$
21	$\{5,3\}_5$	$\{3,5\}_5$	$\{5,3,5\}$	22	$\{5,3\}_5$	$\{3,5\}$	$\{5,3,5\}$

Table 1. The Scope of the Classification Problem in Rank 4.

These cases have all been completely analysed, and the results reported in the literature, with the exception of cases 10 to 12, for which a small amount of additional work remains to be done.

3. The Unfinished Cases

In this short section, we consider cases 10 and 11. Case 12 is dual to case 10, and therefore does not warrant separate consideration. As mentioned earlier, polytopes of these types exist (see [17]). Furthermore, the only polytopes with the given facets and vertex figures are in fact the universal polytopes, as was shown in [8]. However, a full analysis of the quotients of these polytopes has not been done. It will be useful at this point to review the classification of the quotients of the cube (and therefore the octahedron), given in [6]. Let the group of the cube be $W' = \langle s_0, s_1, s_2\rangle$, and let $x = s_0$, $y = s_1 x s_1$ and $z = s_2 y s_2$. The semisparse subgroups of W' are the trivial subgroup

[1] The cube and octahedron have other quotients also, but these all have some degeneracy, and none are projective. See the leading paragraph of Section 3.

(leading to the cube itself), the group $\langle xy \rangle$ and its conjugates $\langle yz \rangle$ and $\langle xz \rangle$ (leading to the digonal prism, that is, a prism whose base is a digon), the group $\langle xyz \rangle$ (leading to the hemicube), and the group $\langle xy, yz \rangle$ (leading to the polytope $\{2,3\}$).

In particular, note that the 3-hemicube (and therefore the 3-hemicross) has no proper quotients (this is not true for higher ranks). Since the vertex figures of $\{\{4,3\},\{3,4\}_3\}$ have no proper quotients, Theorem 2.7 of [9] may be applied to case 10, so that semisparse subgroups of $W = \langle s_0, s_1, s_2, s_3 \rangle = [\{4,3\},\{3,4\}_3]$ are characterised by the property that all their conjugates intersect $\langle s_0, s_1, s_2 \rangle \langle s_1, s_2, s_3 \rangle$ in a semisparse subgroup of W'. It is simple enough (although a little unsatisfying) to do a computer search (for example using [4]) of the subgroups of W to find all those which satisfy the required property. This leads to the following theorem and corollary.

Theorem 3.1. *The universal polytope $\mathcal{P} = \{\{4,3\},\{3,4\}_3\}$ has four quotients. These are $\mathcal{P}$ itself, $\{\{4,3\}_3,\{3,4\}_3\}$, $\{\{2,3\},\{3,4\}_3\}$, and a nonregular polytope whose facets are all digonal prisms.*

Note that the four quotients of $\mathcal{P}$ correspond very naturally to the four quotients of the cube. Note also the following corollary.

Corollary 3.2. *The polytope $\mathcal{Q} = \{\{4,3\}_3,\{3,4\}_3\}$ has no proper quotients.*

The author notes that it would be a worthwhile endeavour to characterize the quotients of $\{\{4,3^{n-3}\},\{3^{n-3},4\}/2\}$ for general $n \geq 4$, but feels that such a characterization falls beyond the scope of this article.

4. The Classification in Rank 4

This final section of the article will go through the cases of Table 1 one by one, giving a description of the universal polytope and its quotients, with references to the literature.

- Cases 1 to 5: The results of [6], where polytopes of "finite type" were studied (that is, polytopes with the same Schläfli symbols as the classical spherical polytopes) show that there are no locally projective polytopes of types $\{3,3,4\}$, $\{3,3,5\}$ or $\{3,4,3\}$. This likewise eliminates cases 9 and 16, as is noted below.
- Case 6: There is no polytope with icosahedral facets and hemidodecahedral vertex figures. See the dual case 8 for an explanation.

- Case 7 is Coxeter and Grünbaum's 11-cell (see [3] or [5]). The polytope has a group of order 660, isomorphic to the projective special linear group $L_2(11)$. It was noted in [7] that this polytope has no proper quotients.
- Case 8: There is no polytope with hemi-icosahedral facets and dodecahedral vertex figures. If such a polytope existed, it could be constructed (in principle) by arranging hemi-icosahedra face to face, with three around each edge, either until the polytope closed up, or ad infinitum. However, an attempt to perform this construction results in the 11-cell of case 7, as noted in [5].
- Case 9 is dual to case 1, thus yields no polytopes.
- Case 10: The universal polytope $\{\{4,3\},\{3,4\}_3\}$ appeared in [17]. A simple proof that the universal polytope is the only polytope of type $\{\{4,3\},\{3,4\}_3\}$ appeared in [8]. The polytope may be constructed from its vertex figures via the "twisting" operation developed by Schulte and McMullen and described in Chapter 9 of [14]. Here, $\{\{4,3\},\{3,4\}_3\}$ is $2^{\mathcal{H}}$, where $\mathcal{H} = \{3,4\}_3$ is the hemicross. Its group is the semidirect product $2^3 \rtimes S_4$, of order 192, with S_4 (the group of $\mathcal{H}$) acting by conjugation on the generators of 2^3 in the same way that it acts on the vertices of the hemicross. The quotients of this polytope were described in Theorem 3.1.
- Case 11: The universal polytope $\{\{4,3\}_3,\{3,4\}_3\}$ is a quotient of $\{\{4,3\},\{3,4\}_3\}$ and has itself no proper quotients. This was shown in Corollary 3.2, but could easily have been noted from the results of [8]. Its group is the quotient of $2^3 \rtimes S_4$ by the (normal) subgroup generated by the product of the three generators of 2^3.
- Case 12 is dual to case 10.
- Case 13 was studied in depth in [7], where it was shown that the universal $\{\{4,3\},\{3,5\}_5\}$ exists and is finite. Its existence and finiteness are also noted in Section 8E of [14]. It appears there as $2^{\mathcal{I}}$, where $\mathcal{I}$ is the hemi-icosahedron (whose group is A_5). Its group is therefore $2^6 \rtimes A_5$, of order 3840, with A_5 acting by conjugation on the generators of 2^6 in the same way that it acts on the vertices of $\mathcal{I}$. The universal polytope therefore has 80 cubes as facets, and 64 vertices. In [7] it was shown, via a computer search, that the polytope has 70 quotients. Of these, three are regular, these being $\{\{2,3\},\{3,5\}_5\}$, the universal polytope $\{\{4,3\},\{3,5\}_5\}$ itself, and a quotient of the latter by a group of order 2. Besides the latter two, there are another nine which are nonregular polytopes of type $\{\{4,3\},\{3,5\}_5\}$, and besides these another eight equivelar polytopes of type $\{4,3,5\}$. The latter eight each have, as facets, some cubes and some hemicubes, hence they are not section regular. For

more details on the quotients of this polytope, the reader is referred to [7].
- Cases 14 and 15: It was shown in Theorem 3.6 of [8] that a polytope with hemicubes for facets cannot be of type $\{4,3,\ldots,3,p\}$ for odd p. Thus cases 14 and 15 do not yield examples of locally projective polytopes.
- Case 16 is dual to case 2.
- Cases 17, 18 and 19 are dual to cases 15, 14 and 13 respectively.
- Case 20 was examined in [10]. The polytope $\{\{5,3\},\{3,5\}_5\}$ exists and is finite, with group $J_1 \times L_2(19)$ of order 600415200. It therefore has 10006920 vertices and half that many facets. Here, $L_2(19)$ is a projective special linear group of order 3420, and J_1 is the first Janko group, of order 175560. The quotients of this polytope are 137 in number, including as regular quotients Coxeter's 57-cell $\{\{5,3\}_5,\{3,5\}_5\}$ (see below), and a polytope of type $\{\{5,3\},\{3,5\}_5\}$ with group J_1. There are also 63 nonregular quotients of this type. The remaining 71 proper quotients (all of type $\{5,3,5\}$) each have, as facets, some dodecahedra and some hemidodecahedra. For more details, see [10] and [11].
- Case 21 was discovered by Coxeter in [2]. The universal polytope has 57 facets, and its group is $L_2(19)$ of order 3420. It was shown in [7] that it has no proper quotients. It is itself a quotient of the universal polytope of case 20.
- Case 22 is dual to case 20, and therefore does not warrant separate discussion.

To summarize, there are nine nondegenerate universal locally projective regular polytopes of rank 4, namely $\{\{3,5\}_5,\{5,3\}_5\}$, $\{\{4,3\},\{3,4\}_3\}$ and its dual, $\{\{4,3\}_3,\{3,4\}_3\}$, $\{\{4,3\},\{3,5\}_5\}$ and its dual, $\{\{5,3\}_5,\{3,5\}_5\}$ and $\{\{5,3\},\{3,5\}_5\}$ and its dual. There are 13 regular polytopes of these types, since each of those with hemi-icosahedral vertex figures (or hemidodecahedral facets) has a proper regular quotient of the same type.

These universal polytopes have a total of 429 quotients, 165 of these being section regular, including 17 regular locally projective polytopes. The extra four regular quotients are the quotients $\{\{2,3\},\mathcal{K}\}$ of locally projective $\{\{4,3\},\mathcal{K}\}$ and their duals. These are also counted amongst the section regular quotients. The 148 nonregular section regular quotients (or their duals) all have hemi-icosahedral vertex figures and spherical (cubic or dodecahedral) facets. Besides these, there are a further four degenerate locally projective polytopes, namely $\{\{2,4\},\{4,3\}_3\}$ and $\{\{2,5\},\{5,3\}_5\}$ and their duals. These four are not quotients of any nondegenerate locally projective polytopes. In summary then, there are 21 regular locally projective polytopes of rank 4, eight degenerate, 13 nondegenerate.

The most remarkable feature of the classification in rank 4 is the following result.

Theorem 4.1. *All locally projective rank 4 polytopes are finite.*

5. The Rank 5 Scope and Unfinished Cases

As for rank 4, to classify the locally projective polytopes in rank 5, we need first to characterise the spherical or projective polytopes that can make up their facets and vertex figures. There are 6 spherical rank 4 polytopes, as discovered by Schläfli [16]. They are the simplex $\{3,3,3\}$, the cube $\{4,3,3\}$ and its dual, the 24-cell $\{3,4,3\}$, and the 120-cell $\{5,3,3\}$ and its dual. Of these, all but the simplex have a projective quotient. Table 2 is similar in structure to Table 1, in that it lists, ignoring duality, all the cases that need to be considered in rank 5.

#	Facets	V.Figures	Type	#	Facets	V.Figures	Type
1	$\{3,3,3\}$	$\{3,3,4\}_4$	$\{3,3,3,4\}$	2	$\{3,3,3\}$	$\{3,3,5\}_{15}$	$\{3,3,3,5\}$
3	$\{3,3,4\}$	$\{3,4,3\}_6$	$\{3,3,4,3\}$	4	$\{3,3,4\}_4$	$\{3,4,3\}_6$	$\{3,3,4,3\}$
5	$\{3,3,4\}_4$	$\{3,4,3\}$	$\{3,3,4,3\}$	6	$\{3,4,3\}$	$\{4,3,3\}_4$	$\{3,4,3,3\}$
7	$\{3,4,3\}_6$	$\{4,3,3\}_4$	$\{3,4,3,3\}$	8	$\{3,4,3\}_6$	$\{4,3,3\}$	$\{3,4,3,3\}$
9	$\{4,3,3\}_4$	$\{3,3,3\}$	$\{4,3,3,3\}$	10	$\{4,3,3\}$	$\{3,3,4\}_4$	$\{4,3,3,4\}$
11	$\{4,3,3\}_4$	$\{3,3,4\}_4$	$\{4,3,3,4\}$	12	$\{4,3,3\}_4$	$\{3,3,4\}$	$\{4,3,3,4\}$
13	$\{4,3,3\}$	$\{3,3,5\}_{15}$	$\{4,3,3,5\}$	14	$\{4,3,3\}_4$	$\{3,3,5\}_{15}$	$\{4,3,3,5\}$
15	$\{4,3,3\}_4$	$\{3,3,5\}$	$\{4,3,3,5\}$	16	$\{5,3,3\}_{15}$	$\{3,3,3\}$	$\{5,3,3,3\}$
17	$\{5,3,3\}$	$\{3,3,4\}_4$	$\{5,3,3,4\}$	18	$\{5,3,3\}_{15}$	$\{3,3,4\}_4$	$\{5,3,3,4\}$
19	$\{5,3,3\}_{15}$	$\{3,3,4\}$	$\{5,3,3,4\}$	20	$\{5,3,3\}$	$\{3,3,5\}_{15}$	$\{5,3,3,5\}$
21	$\{5,3,3\}_{15}$	$\{3,3,5\}_{15}$	$\{5,3,3,5\}$	22	$\{5,3,3\}_{15}$	$\{3,3,5\}$	$\{5,3,3,5\}$

Table 2. The Scope of the Classification Problem in Rank 5.

A number of these cases have been dealt with in the literature, in the sense that it is known whether or not the universal polytope exists and is finite. Of the remainder, duality again reduces the work significantly, so only cases 2, 5, 20 and 21 need to be analysed here.

We start with the easy case first.

Theorem 5.1. *The universal polytope* $\{\{3,3,4\}_4,\{3,4,3\}\}$ *does not exist.*

Proof. In fact, if this polytope exists it must be finite. Since its facets are flat, with 4 vertices, the polytope itself must be flat, with 4 vertices. Each vertex figure has 1152 flags, so the total number of flags is 4608.

However, analysing (using GAP version 4, see [4]) the group $\langle s_0, \dots, s_4\rangle$ of $\{3,3,4,3\}$ with the extra relator $(s_0s_1s_2s_3)^4$ imposed reveals that its order is only 2304. ∎

What is happening here is that the relator $(s_0s_1s_2s_3)^4$ implies also the relator $(s_1s_2s_3s_4)^6$, so that the polytope collapses to the $\{\{3,3,4\}_4,\{3,4,3\}_6\}$.

The next two results will rely on the concept of an *abelian invariant* of a group. For a group G, the *derived group* is the group $G' = \langle ghg^{-1}h^{-1}\rangle$ generated by all commutators of G. This will be a normal subgroup of G, and the quotient G/G' will be an abelian group, in some sense the largest abelian quotient of G (see, for example, Theorem 2.23 of [15], or any good group theory reference). GAP Version 4 (see [4]) has algorithms to quickly find the structure of this largest abelian quotient. If the quotient is infinite, it proves, of course, that the original group G is also infinite.

The results will also use the concept of the *core* of a subgroup G of a group W. The core of G in W is the intersection of all conjugates G^w of G by elements of W. It will be a normal subgroup of both G and W.

Theorem 5.2. *The universal polytope* $\mathcal{P}=\{\{3,3,3\},\{3,3,5\}_{15}\}$ *exists and is infinite.*

Proof. Let $W = \langle a,b,c,d,e\rangle$ be the group $[3,3,3,5]$, with additional relator $(bcde)^{15}$, that is, W is the automorphism group of $\mathcal{P}$, if it exists. Using GAP, it was discovered that this group has a subgroup $G = \langle e,d,(cdede)^2c, e^{dcbaededcb}c, a(a^{bcdedcbedcbeded}b)^{cba}, (ca)^{bcdedcbededcdedcedbc}\rangle$ of index 3360, for which $G/G' \cong \mathbf{Z}_2^4 \times \mathbf{Z}$. Since therefore G is infinite, so is W, so that if the universal polytope $\mathcal{P}$ exists, it is infinite.

The group G has a core N of index 1958400 in W. GAP may be used again to find the quotient $W/N = \langle aN,bN,cN,dN,eN\rangle$ and verify that it is a string C-group, and therefore the group of a polytope, with facets $\{3,3,3\}$ and vertex figures $\{3,3,5\}_{15}$. Since there exist polytopes of the type being considered, the universal polytope exists. ∎

This group W/N is isomorphic to $2 \times S_4(4)$, where $S_4(4)$ is a symplectic group, a simple group of order 979200. The polytope has a proper regular quotient $(W/N)/2$, isomorphic to W/N' say, whose group is exactly $S_4(4)$. In fact, this polytope could have been constructed by taking the quotient of $\mathcal{P}$ by the core N' of the index 85 group $\langle a,c,e,bcab,dedecd\rangle \leq W$.

The proof of the next theorem is very similar to that of the last.

Theorem 5.3. *The universal polytope* $\mathcal{P} = \{\{5,3,3\}_{15},\{3,3,5\}_{15}\}$ *exists and is infinite.*

Proof. Let $W=\langle a,b,c,d,e\rangle$ be the group $[5,3,3,5]$, with additional relators $(abcd)^{15}$ and $(bcde)^{15}$, that is, W is the automorphism group of $\mathcal{P}$, if it exists. Using GAP, it was discovered that this group has a subgroup $G=\langle a,c,e,badcedb,dcbabaededcb\rangle$ of index 75, for which $G/G'\cong \mathbf{Z}_2^2\times\mathbf{Z}$. Since therefore G is infinite, so is W, so that if the universal polytope $\mathcal{P}$ exists, it is infinite.

The group G has a core N of index 39980092542088642560 in W. Again GAP may be used to find the quotient $W/N=\langle aN,bN,cN,dN,eN\rangle\cong 3^{25}\rtimes(2^{16}\rtimes H)$ (here H is the group of the hemi-120-cell, that is, $(A_5\times A_5)\rtimes 2$). GAP may be used yet again to verify that the group is a string C-group, and therefore the group of a polytope, with facets $\{5,3,3\}_{15}$ and vertex figures $\{3,3,5\}_{15}$. Since there exist polytopes of the type being considered, the universal polytope exists. ∎

In fact, there exist other normal subgroups of W. For one in particular, the quotient W/N' is a string C-group of order 471859200, namely, the factor $2^{16}\rtimes H$ of the group mentioned in the theorem. This N' may be constructed as the core, for example, of $\langle a,c,e,bacbab,dedecd,a^{bcdab}\rangle$. Write W/N' as $W/N'=W'=\langle s_0,s_1,s_2,s_3,s_4\rangle$. This W' is the group of a self-dual polytope of type $\{5,3,3,5\}$ with $65536=2^{16}$ facets $\{5,3,3\}_{15}$. Let $\omega=(s_2s_3s_4)^5$. This ω is a central involution of the parabolic subgroup $H_1=\langle s_0,s_2,s_3,s_4\rangle$ of W', and we may attempt the generalised petrial construction described in [12]. Doing so, by writing $W''=\langle s_0\omega,s_1,s_2,s_3,s_4\rangle$, yields another string C-group W'', isomorphic to W'. The corresponding polytope is of type $\{4,3,3,5\}$, with hypercube facets, and projective vertex figures of type $\{3,3,5\}_{15}$, and is therefore locally projective. In fact, it is one of the two finite polytopes of this type given at the end of [8].

The same construction applied to the polytope of Theorem 5.3 with the group $W/N\cong 3^{25}\rtimes W'$ yields a polytope of type $\{12,3,3,5\}$ with projective vertex figures. The facets are evidently neither spherical or projective. In fact, the facet is a non-universal polytope whose symmetry group is solvable of order $2^7 3^{12}$.

The last case to be considered is not quite completed. It is not known whether or not the polytope exists. However, it is known that it cannot be finite.

Corollary 5.4. *If the universal polytope* $\mathcal{P}=\{\{5,3,3\},\{3,3,5\}_{15}\}$ *exists, it is infinite.*

Proof. If it exists, it covers $\{\{5,3,3\}_{15},\{3,3,5\}_{15}\}$, so is certainly infinite. ∎

To prove the existence of this polytope, it would be enough to show a single example of a polytope with facets of type $\{5,3,3\}$ and vertex figures of type $\{3,3,5\}_{15}$. To prove nonexistence it would have to be shown that imposing the extra relator $(s_1s_2s_3s_4)^{15}$ on the group $[5,3,3,5]$ either caused it to fail the intersection property, or caused it to collapse to the group of the $\{\{5,3,3\}_{15},\{3,3,5\}_{15}\}$.

6. The Rank 5 Classification

Here, we do not attempt to enumerate the quotients of the polytopes found. Instead, the cases of Table 2 will be mentioned one by one, and it will be stated, with appropriate references, whether or not the universal polytope exists, and whether or not it is finite.

- Case 1: The results of [6] or Theorem 3.6 of [8] show that there are no locally projective polytopes of type $\{4,3,3,3\}$.
- Case 2: By Theorem 5.2 this exists and is infinite.
- Cases 3 and 4: These were examined in [13] (see also Section 14A of [14]), where it was noted that both exist and are finite. The $\{\{3,3,4\},\{3,4,3\}_6\}$, denoted $\mathcal{L}_0^5$ in [13], has group of order 9216. It is a four-fold cover of the universal $\mathcal{L}_3^5 = \{\{3,3,4\}_4,\{3,4,3\}_6\}$.
- Case 5: By Theorem 5.1 the $\{\{3,3,4\}_4,\{3,4,3\}\}$ does not exist.
- Cases 6 to 9: Cases 6, 7, 8 and 9 are dual to cases 5, 4, 3 and 1 respectively.
- Cases 10 to 12: These exist, and are the only polytopes of their types, as was shown in Theorem 4.5 of [8]. The $\{\{4,3,3\},\{3,3,4\}_4\}$ has group $2^4 \rtimes H$, of order 3072 (as does its dual), where H is the group of the hemi-cross-polytope $\{3,3,4\}_4$, acting by conjugation on the generators of 2^4 in the same way that it acts on the vertices of the hemi-cross-polytope. It is a two-fold cover of $\{\{4,3,3\}_4,\{3,3,4\}_4\}$. Note that Section 13D of [14] gives a family of sets of five involutions $R_{-1},\dots,R_3$, and asks when they generate the group of a polytope. One member of this family has $\langle R_{-1},\dots,R_2\rangle$ generating the group of $\{4,3,3\}_4$, and $\langle R_0,\dots,R_3\rangle$ the group of $\{3,3,4\}_4$. At the end of the section, the authors note correctly, but for invalid reasons, that $\langle R_{-1},\dots,R_3\rangle$ is not the group of a polytope.
- Case 13: The $\{\{4,3,3\},\{3,3,5\}_{15}\}$ was considered in section 5 of [8]. It was shown that this polytope exists, is infinite, and has infinitely many finite quotients. Three examples of such finite quotients were illustrated, one of which was mentioned in this article in the notes following Theorem 5.3.
- Cases 14 and 15: Theorem 3.6 of [8] shows that there are no polytopes of type $\{4,3,3,5\}$ whose facets are hemicubes.
- Cases 16 to 19: Cases 16, 17, 18 and 19 are dual to cases 2, 15, 14 and 13 respectively, and so do not warrant separate discussion.

- Case 20: This may or may not exist. If it exists, it is infinite. See Corollary 5.4.
- Case 21: This exists and is infinite, as shown in Theorem 5.3.
- Case 22: As for its dual, case 20, it is not known whether or not this polytope exists. However, it is not finite.

In summary, the 22 cases give rise to either 12 or 14 rank 5 universal locally projective polytopes, including 7 finite universal locally projective polytopes. Amongst their quotients lie infinitely many non-universal regular locally projective polytopes. Each of the five projective rank 4 polytopes $\mathcal{K}$ gives rise to a further two degenerate finite universal locally projective polytopes, namely $\{2,\mathcal{K}\}$ and $\{\mathcal{K},2\}$. Only four of these ten degenerate polytopes lie among the quotients of the non-degenerate ones.

7. The Classification in Rank $n \geq 6$

The classification in rank $n \geq 6$ has been completed. The only projective rank 5 or higher polytopes are the hemicubes and their duals. Hence, the only cases that need to be considered are those listed in Table 3.

#	Facets	V.Figures	Type
1	$\{3,3,\dots,3\}$	$\{3,\dots,3,4\}_{n-1}$	$\{3,3,\dots,3,4\}$
2	$\{4,3,\dots,3\}_{n-1}$	$\{3,\dots,3,3\}$	$\{4,3,\dots,3,3\}$
3	$\{4,3,\dots,3\}$	$\{3,\dots,3,4\}_{n-1}$	$\{4,3,\dots,3,4\}$
4	$\{4,3,\dots,3\}_{n-1}$	$\{3,\dots,3,4\}_{n-1}$	$\{4,3,\dots,3,4\}$
5	$\{4,3,\dots,3\}_{n-1}$	$\{3,\dots,3,4\}$	$\{4,3,\dots,3,4\}$

Table 3. The Scope of the Classification Problem in Rank n.

Cases 1 and 2 were dealt with in Theorem 2.14 of [6], or in Theorem 3.6 of [8]. There are no locally projective polytopes of type $\{3,3,\dots,3,4\}$ or $\{4,3,\dots,3,3\}$. Cases 3 to 5 all exist, and are the only locally projective polytopes of their types. This was shown in Theorem 4.5 of [8]. The polytope $\mathcal{P}=\{\{4,3,\dots,3\},\{3,\dots,3,4\}_{n-1}\}$ has group $2^{n-1}\rtimes H$, of order $2^{2n-3}\cdot(n-1)!$, where H is the group of the hemi-cross-polytope $\mathcal{H}$, acting on the generators of 2^{n-1} in the same way it acts on the vertices of $\mathcal{H}$. The universal $\mathcal{P}$ is a two-fold cover of the universal $\{\{4,3,\dots,3\}_{n-1},\{3,\dots,3,4\}_{n-1}\}$.

8. Conclusion and Summary

The locally projective universal polytopes have been classified, and have been found to be finite except for certain rank 5 cases. Here, polytopes are defined to be locally projective if their facets and vertex figures are spherical or projective but not both projective. Furthermore, all quotients of the rank 4 locally projective polytopes have been enumerated. It should be noted that the problem of classifying polytopes whose minimal nonspherical sections are projective would likely be extremely difficult, since it would require first classifying at least the regular quotients of the infinite rank 5 universal locally projective polytopes of type $\{4,3,3,5\}$ and $\{3,3,3,5\}$.

Table 4 lists the universal locally projective polytopes, along with some summary information. Duals are not listed seperately.

Rank	Facets	V.Figures	Aut($\mathcal{P}$)	$\lvert$Aut($\mathcal{P}$)$\rvert$	#Facets	#Vertices
4	$\{2,4\}$	$\{4,3\}_3$	$2\times S_4$	48	3	2
4	$\{2,3\}$	$\{3,4\}_3$	$2\times S_4$	48	4	2
4	$\{2,5\}$	$\{5,3\}_5$	$2\times A_5$	120	6	2
4	$\{2,3\}$	$\{3,5\}_5$	$2\times A_5$	120	10	2
4	$\{4,3\}_3$	$\{3,4\}_3$	$(2^3\rtimes S_4)/2$	96	4	4
4	$\{4,3\}$	$\{3,4\}_3$	$2^3\rtimes S_4$	192	4	8
4	$\{3,5\}_5$	$\{5,3\}_5$	$L_2(11)$	660	11	11
4	$\{5,3\}_5$	$\{3,5\}_5$	$L_2(19)$	3420	57	57
4	$\{4,3\}$	$\{3,5\}_5$	$2^6\rtimes A_5$	3840	80	64
4	$\{5,3\}$	$\{3,5\}_5$	$J_1\times L_2(19)$	600415200	5003460	10006920
5	$\{2,4,3\}$	$\{4,3,3\}_3$		384	4	2
5	$\{2,3,3\}$	$\{3,3,4\}_3$		384	8	2
5	$\{2,3,4\}$	$\{3,4,3\}_6$		1152	12	2
5	$\{2,5,3\}$	$\{5,3,3\}_{15}$		14400	60	2
5	$\{2,3,3\}$	$\{3,3,5\}_{15}$		14400	300	2
5	$\{4,3,3\}_4$	$\{3,3,4\}_4$	see text	1536	8	8
5	$\{3,3,4\}_4$	$\{3,4,3\}_6$		2304	12	4
5	$\{4,3,3\}$	$\{3,3,4\}_4$	see text	3072	8	16
5	$\{3,3,4\}$	$\{3,4,3\}_6$		9216	24	16
5	$\{3,3,3\}$	$\{3,3,5\}_{15}$		∞	∞	∞
5	$\{4,3,3\}$	$\{3,3,5\}_{15}$	see [8]	∞	∞	∞
5	$\{5,3,3\}_{15}$	$\{3,3,5\}_{15}$		∞	∞	∞
5	$\{5,3,3\}$	$\{3,3,5\}_{15}$		∞[2]	∞	∞
n	$\{2,4,3,\ldots,3\}$	$\{4,3,\ldots,3\}_{n-1}$		$2^{n-1}(n-1)!$	$n-1$	2
n	$\{2,3,3,\ldots,3\}$	$\{3,\ldots,3,4\}_{n-1}$		$2^{n-1}(n-1)!$	2^{n-2}	2
n	$\{4,3,\ldots,3\}_{n-1}$	$\{3,\ldots,3,4\}_{n-1}$	see text	$2^{2n-4}(n-1)!$	2^{n-2}	2^{n-2}
n	$\{4,3,\ldots,3\}$	$\{3,\ldots,3,4\}_{n-1}$	see text	$2^{2n-3}(n-1)!$	2^{n-2}	2^{n-1}

Table 4. The Locally Projective Universal Polytopes.

It is not surprising that the locally projective polytopes of type $\{3,3,4,3\}$ and $\{4,3,\ldots,3,4\}$ should prove to be finite. Consider what happens topo-

[2] If it exists at all.

logically when one facet of a tesselation is made projective. The centroid of the facet becomes a point mirror, and other points are identified with their reflections through this mirror. One could arbitrarily define a hyperplane passing through this mirror, and represent the quotient space as one of its halfspaces, with suitable identifications of its boundary points. It is difficult to see how this process could be repeated for each facet in a euclidean tesselation, still maintaining an infinite quotient space. It is much easier to see how a hyperbolic space could be remain infinite despite being "pinched" by an infinite number of point mirrors, provided the mirrors were all sufficiently far apart.

It seems that in the hyperbolic tesselations $\{3,5,3\}$, $\{5,3,4\}$ and $\{5,3,5\}$, the centroids of the facets are not sufficiently far apart, whereas they are in $\{5,3,3,3\}$, $\{5,3,3,4\}$ and $\{5,3,3,5\}$. It is interesting to speculate whether the finiteness of the locally projective rank 4 polytopes is an "intrinsic feature" (in some sense) of four-dimensional hyperbolic space, or just some sort of platonic fluke.

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